ON THE LIE ALGEBRAS OF SURFACE PURE BRAID GROUPS

B. ENRIQUEZ AND V. V. VERSHININ

Abstract. We consider the Lie algebra associated with the descending central series filtration of the pure braid group of a closed surface of arbitrary genus. R. Bezrukavnikov gave a presentation of this Lie algebra over the rational numbers. We show that his presentation remains true for this Lie algebra itself, i. e. over integers.

Contents

1. Introduction

Let $S_{g,b}$ be an oriented surface of genus g with b boundary components. We denote by $Br_n(S_{g,b})$ the braid group on n strings of the surface $S_{g,b}$ and by $P_n(S_{g,b})$ the pure braid group on n strings of the surface $S_{g,b}$. There exists an exact sequence:

$$1 \to P_n(S_{g,b}) \to Br_n(S_{g,b}) \to \Sigma_n \to 1,$$

where $Br_n(S_{g,b}) \to \Sigma_n$ is a natural epimorphism to the symmetric group Σ_n, so, $P_n(S_{g,b})$ is its kernel. The classical braid groups are the braid groups of a disc: $Br_n \cong Br_n(S_{0,1})$, $P_n \cong P_n(S_{0,1})$.

In the paper we consider the pure braid group of a closed orientable surface of genus g with no boundary components, $P_n(S_{g,0})$, which we denote for simplicity by $P_n(S_{g,})$

Usually the braid group Br_n is given by the following Artin presentation [1]. It has the generators σ_i, $i = 1, ..., n-1$, and the two types of relations:

$$\begin{cases} \sigma_i\sigma_j &= \sigma_j\,\sigma_i, \quad \text{if} \quad |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}\,. \end{cases} \tag{1.1}$$

The generators $a_{i,j}$, $1 \le i < j \le n$ for the pure braid group P_n (of a disc) can be defined (as elements of the the braid group Br_n) by the formula:

$$a_{i,j} = \sigma_{j-1}...\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}...\sigma_{j-1}^{-1}.$$

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Then the defining relations, which are called the *Burau relations* [3], [13] are as follows:

$$\begin{cases} a_{i,j}a_{k,l} = a_{k,l}a_{i,j} \text{ for } i<j<k<l \text{ and } i<k<l<j, \\ a_{i,j}a_{i,k}a_{j,k} = a_{i,k}a_{j,k}a_{i,j} \text{ for } i<j<k, \\ a_{i,k}a_{j,k}a_{i,j} = a_{j,k}a_{i,j}a_{i,k} \text{ for } i<j<k, \\ a_{i,k}a_{j,k}a_{j,l}a_{j,k}^{-1} = a_{j,k}a_{j,l}a_{j,k}^{-1}a_{i,k} \text{ for } i<j<k<l. \end{cases} \tag{1.2}$$

It was proved by O. Zariski [17] and then rediscovered by E. Fadell and J. Van Buskirk [5] that a presentation for the braid group of a sphere can be given with the generators σ_i, $i = 1, ..., n-1$, the same as for the classical braid group, satisfying the braid relations (1.1) and the following sphere relation:

$$\sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \dots \sigma_2\sigma_1 = 1.$$

For the pure braid group on a sphere let us introduce the elements $a_{i,j}$ for all i, j by the formulae:

$$\begin{cases} a_{j,i} = a_{i,j} \quad \text{for } i<j\leq n, \\ a_{i,i} = 1. \end{cases} \tag{1.3}$$

The pure braid group for the sphere has the generators $a_{i,j}$ which satisfy Burau relations (1.2), relations (1.3), and the following relations [7]:

$$a_{i,i+1}a_{i,i+2} \dots a_{i,i+n-1} = 1 \quad \text{for all } i \leq n,$$

where $k+n$ is considered modulo n having in mind (1.3).

For a group G the descending central series

$$G = \Gamma_1 > \Gamma_2 > \dots > \Gamma_i > \Gamma_{i+1} > \dots .$$

is defined by the formulas

$$\Gamma_1 = G, \quad \Gamma_{i+1} = [\Gamma_i, G].$$

It gives rise to the associated graded Lie algebra (over $\mathbb{Z}$) $gr^*(G)$ [16]:

$$gr^i(G) = \Gamma_i/\Gamma_{i+1}.$$

A presentation of the Lie algebra $gr^*(P_n)$ for the pure braid group can be described as follows [12]. It is the quotient of the free Lie algebra $L[A_{i,j}| \, 1 \leq i < j \leq n]$, generated by elements $A_{i,j}$ with $1 \leq i < j \leq n$, modulo the "infinitesimal braid relations" or "horizontal $4T$ relations" given by the following three relations:

$$\begin{cases} [A_{i,j}, A_{s,t}] = 0, \text{ if } \{i,j\} \cap \{s,t\} = \phi, \\ [A_{i,j}, A_{i,k} + A_{j,k}] = 0, \text{ if } i<j<k, \\ [A_{i,k}, A_{i,j} + A_{j,k}] = 0, \text{ if } i<j<k. \end{cases}$$

A similar presentation of the Lie algebra of descending central series filtration of the upper triangular McCool group is obtained in [4].

Y. Ihara in [9] gave a presentation of the Lie algebra $gr^*(P_n(S^2))$ of the pure braid group of the sphere. It is convenient to have conventions like (1.3). Hence, $gr^*(P_n(S^2))$ is the quotient

of the free Lie algebra $L[B_{i,j}| 1 \leq i,j \leq n]$ generated by elements $B_{i,j}$ with $1 \leq i,j \leq n$ modulo the following relations:

$$\begin{cases} B_{i,j} = B_{j,i} \text{ for } 1 \leq i,j \leq n, \\ B_{i,i} = 0 \ \text{ for } 1 \leq i \leq n, \\ [B_{i,j}, B_{s,t}] = 0, \text{ if } \{i,j\} \cap \{s,t\} = \phi, \\ \sum_{j=1}^{n} B_{i,j} = 0, \text{ for } 1 \leq i \leq n. \end{cases} \tag{1.4}$$

Also a presentation of the Lie algebra $gr^*(P_n(S^2))$ can be given with generators $A_{i,j}$ with $1 \leq i < j \leq n-1$, modulo the following relations [11]:

$$\begin{cases} [A_{i,j}, A_{s,t}] = 0, \text{ if } \ \{i,j\} \cap \{s,t\} = \phi, \\ 2(\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} A_{i,j}) = 0. \end{cases}$$

So, the element $\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} A_{i,j}$ of order 2 generates the central subalgebra in $gr^*(P_n(S^2))$.

We study the natural Lie algebra obtained from the descending central series for $(P_n(S_g)$ for $g \geq 1$, for $g = 0$ the structure of $gr^*(P_n(S^2))$ is given by the result of Y. Ihara (1.4).

The essential ingredient in the study of the pure braid groups is a natural fibration of configuration spaces and its initial term of the homotopy exact sequence. Let $F(S_g, n)$ be the space of n-tuples of pairwise different points in S_g, then we have the following fibration:

$$S_g \setminus Q_{n-1} \to F(S_g, n) \to F(S_g, n-1),$$

where $S_g \setminus Q_{n-1}$ is the surface with $n-1$ points deleted and $F(S_g, n) \to F(S_g, n-1)$ is the projection on the first $n-1$ components of an n-tuple; $S_g \setminus Q_{n-1}$ is a fiber of the fibration. This fibration generates the exact sequence of groups:

$$1 \to \pi_1(S_g \setminus Q_{n-1}) \to P_n(S_g) \to P_{n-1}(S_g) \to 1, \tag{1.5}$$

where $\pi_1(S_g \setminus Q_{n-1})$ is a $(g, n-1)$-*surface group*; it is a free group on $2g+n-2$ generators and it has the canonical presentation

$$\pi_{g,n-1} = \pi_1(S_g \setminus Q_{n-1}) =< a_1, c_1, \ldots, a_g, c_g, u_1, \ldots, u_{n-1} \mid \prod_{i=1}^{n-1} u_i \prod_{m=1}^{g} [a_m, c_m] = 1 > .$$

If we consider the descending central series filtration of the groups of this exact sequence and apply the functor of the associated Lie algebras, then the corresponding sequence will be not left exact for $n \geq 3$:

$$gr^*(\pi_1(S_g \setminus Q_{n-1})) \to gr^*(P_n(S_g)) \to gr^*(P_{n-1}(S_g)) \to 1.$$

To fix the situation another filtration was introduced by M. Kaneko [10] and H. Nakamura and H. Tsunogai [15]. The authors call it the *weight filtration*. Roughly speaking the difference with respect to the descending central series filtration is that the elements $u_1, \ldots, u_{n-1}$ are given the grading two instead of one. For $\pi_{g,0}$ and $\pi_{g,1}$ it coincides with the descending central series filtration.

$$\pi_{g,k}(1) = \pi_{g,k},$$
$$\pi_{g,k}(2) = [\pi_{g,k}, \pi_{g,k}] < u_1, \ldots, u_k >,$$
$$\pi_{g,k}(m) =< [\pi_{g,k}(i), \pi_{g,k}(j)] \mid i+j = m >, m \geq 3.$$

$$P_n(S_g)(1) = P_n(S_g),$$
$$P_n(S_g)(2) = [P_n(S_g), P_n(S_g)] \, \pi_{g,n-1}(2),$$

$$P_n(S_g)(m) =< [P_n(S_g)(i), P_n(S_g)(j)] \mid i + j = m >, m \geq 3.$$

It is proved by H. Nakamura, N. Takao and R. Ueno [14] that the sequence of Lie algebras associated to the weight filtration and corresponding to the sequence (1.5)

$$1 \to gr_w^*(\pi_1(S_g \setminus Q_{n-1})) \to gr_w^*(P_n(S_g)) \to gr_w^*(P_{n-1}(S_g)) \to 1$$

is exact (even for a punctured surface).

2. Lie algebra $gr^*(P_n(S_g))$

R. Bezrukavnikov [2] determined the Lie algebra $gr^*(P_n(S_g)) \otimes \mathbb{Q}$. It has the generators $a_{l,i}$, $b_{l,i}$ $1 \leq l \leq g$, $1 \leq i \leq n$, so l corresponds to the genus of a surface and i corresponds to the number of strings. Relations are as follows

$$(2.1) \quad \begin{cases} [a_{l,i}, b_{k,j}] & = 0 \;\text{ for } i \neq j, \; l \neq k, \\ [a_{l,i}, a_{k,j}] & = 0 \;\text{ for } i \neq j, \\ [b_{l,i}, b_{k,j}] & = 0 \;\text{ for } i \neq j, \\ [a_{l,i}, b_{l,j}] & = [a_{k,j}, b_{k,i}] \;\text{ for all } k, l \text{ and } i \neq j, \text{ denote it by } \; s_{i,j}, \\ \sum_{l=1}^{g} [a_{l,i}, b_{l,i}] & = -\sum_{j \neq i} s_{i,j}, \\ [a_{l,i}, s_{j,k}] & = 0 \;\text{ for } i \neq j \neq k, \\ [b_{l,i}, s_{j,k}] & = 0 \;\text{ for } i \neq j \neq k. \end{cases}$$

The aim of the present work is to prove that this presentation is true over $\mathbb{Z}$.

The Lie algebra associated with the weight filtration for the pure braid group of a punctured surface was described by H. Nakamura, N. Takao and R. Ueno in [14]. For a surface without punctures the description of H. Nakamura, N. Takao and R. Ueno coincides with that of R. Bezrukavnikov.

Theorem 2.1. *For the pure braid group of a closed surface of the genus g, $gr^*(P_n(S_g))$, the descending central series filtration and the weight filtration coincide. The generators $a_{l,i}$, $b_{l,i}$ $1 \leq l \leq g$, $1 \leq i \leq n$, and the Bezrukavnikov relations (2.1) give a presentation of the graded Lie algebra $gr^*(P_n(S_g))$.*

Proof. We denote by Γ_i the descending central series filtration on $P_n(S_g)$. Let us consider the presentation of a surface pure braid group given by D. L. Gonçalves and J. Guaschi [8]. There are $2ng$ generators: $\rho_{i,l}$ and $\tau_{i,l}$, $1 \leq l \leq g$, $1 \leq i \leq n$, and relations of 30 types .

Evidently, we have $\Gamma_1 = P_n(S_g)(1)$, $\Gamma_2 \subset P_n(S_g)(2)$. So, there is a commutative diagram of the exact sequences

$$\begin{array}{ccccccc} 1 & \to \Gamma_2 & \longrightarrow & P_n(S_g) & \longrightarrow & g^1(P_n(S_g)) \to 1 \\ & \psi\Big\downarrow & & \Big\downarrow Id & & \phi\Big\downarrow \\ 1 \to & P_n(S_g)(2) & \longrightarrow & P_n(S_g) & \longrightarrow & g_w^1(P_n(S_g)) \to 1 \end{array}$$

The map ϕ is surjective and the rank of $g_w^1(P_n(S_g))$ is equal to $2gn$. The rank of $g^1(P_n(S_g))$ can not be bigger than $2gn$, so it is an isomorphism. Hence ψ is also an isomorphism. From the definition of $P_n(S_g)(m)$ it follows that $\Gamma_m = P_n(S_g)(m)$.

□

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IRMA (CNRS), rue René Descartes, F-67084 Strasbourg, France
E-mail address: enriquez@@math.u-strasbg.fr

Département des Sciences Mathématiques, Université Montpellier II, Place Eugène Bataillon, 34095 Montpellier cedex 5, France
E-mail address: vershini@math.univ-montp2.fr

Sobolev Institute of Mathematics, Novosibirsk 630090, Russia
E-mail address: versh@math.nsc.ru